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Peierls Argument and Long-Range Order Behavior of Quantum Lattice Systems with Unbounded Spins

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Results on long-range order behavior are obtained for systems in arbitrary dimension ($\nu \geqslant 2$) with a wide class of spin–spin long-range interactions, without assuming the reflection positivity property.

KEY WORDS: Peierls argument; long range behavior; quantum lattice system; unbounded spin.

1. INTRODUCTION

The presence of critical behavior in statistical mechanical systems is indicated by the appearance of the so-called *long-range order*. This means that for two spins x_i and x_j situated at the points i and j, the two-point correlation function $\langle x_i x_j \rangle$ does not tend to zero when the distance between i and j tends to infinity. For quantum lattice systems in dimension $\nu \geqslant 3$ with interaction between nearest neighbour pairs, the appearance of the long-range order was proved in the papers[1–8] using the method of infrared domination. In the papers by Glimm, Jaffe and Spencer[9] such a result was obtained for some two-dimensional continuous systems of Euclidean quantum field theory with the help of the Peierls argument.[10] The lattice approximation of the $P_4(\phi)_2$ model is a classical spin system with unbounded spins situated in the potential field $V(x_j) = P_4(x_j)$, $j \in \mathbf{Z}^2$ (P_4 being polynomial of degree 4) and subject to a two-point nearest neighbours interaction. It was soon realized, in refs. 11 and 3, that the

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method also works in the case of the unbounded spin systems which describe an anharmonic quantum crystals. In refs. 11 and 3 nearest neighbour interaction and reflection positivity are required.

In the present paper we establish the existence of the long-range order for quantum, lattice systems with unbounded spin and any lattice dimension. Our assumptions allow non-nearest neighbours interaction between spins, and do not require reflection positivity. The main tools we use for the description of quantum systems are the Feynman–Kac[12] formula and the technique of functional integrals on the periodic path space.[13,14] The proof of the main technical lemma is based on the probability estimate technique of Ruelle[15] (already used in a related context by Park[16] (see also ref. 17)).

Let us briefly describe the contents of this paper. In Section 2 we define the systems we consider. In Section 3 we define our "Peierls framework" and prove the main theorem. In Section 4 we prove the main technical lemma, which then leads to the existence of quantum states and to our main theorem.

2. DEFINITIONS AND NOTATIONS

Let us consider the ν-dimensional integer lattice $\mathbf{Z}^\nu$. At each site i of the lattice there is an unbounded one-component continuous spin x_i. This means that with each site $i \in \mathbf{Z}^\nu$ we associate a one-particle physical Hilbert space $L^2(\mathbf{R}, dx_i)$, where dx_i is the Lebesgue measure on $\mathbf{R}$.

For each $x_i \in \mathbf{R}$ and each $i = (i^{(1)},..., i^{(\nu)}) \in \mathbf{Z}^\nu$ we define the distance from the origin by the formula:

$$|i| = \max_{1 \leqslant \alpha \leqslant \nu} |i^{(\alpha)}| \tag{2.1}$$

For each bounded region $\Lambda \subset \mathbf{Z}^\nu$ introduce the notations

$$x_\Lambda = \{x_i : i \in \Lambda\}, \qquad dx_\Lambda = \prod_{i \in \Lambda} dx_i$$

$$\mathscr{H}_\Lambda = \bigotimes_{i \in \Lambda} L^2(\mathbf{R}, dx_i) \tag{2.2}$$

and define a local Hamiltonian as given by the operator on $\mathscr{H}_\Lambda$

$$H_\Lambda = -\frac{1}{2} \sum_{i \in \Lambda} \frac{d^2}{dx_i^2} + V(x_\Lambda) \tag{2.3}$$

and

$$V(x_\Lambda)=\lambda\sum_{i\in\Lambda}(x_i^2-\lambda^{-1})^2+\tfrac{1}{2}\lambda\sum_{i\in\Lambda}x_i^2+\tfrac{1}{2}\sum_{(i,j)\in\Lambda}d_{ij}(x_i-x_j)^2 \qquad (2.4)$$

Here λ is a real constant that gives the strength of the interaction, $d_{ij}=d_{|i-j|}$ is a function on $\mathbf{Z}^\nu$ and sum over (i,j) means sum over all possible pairs of sites in Λ. For the function d_{ij} we shall assume that there exist some constants d, D and $\varepsilon>0$ such that

$$\begin{aligned} &d_{ij}\geqslant d>0 \qquad \text{for} \quad |i-j|=1\\ &0\leqslant d_{ij}\leqslant D\,|i-j|^{-\nu-\varepsilon}\end{aligned} \qquad (2.5)$$

The contraction of the local Hamiltonian in the region $\Lambda\subset\mathbf{Z}^\nu$ corresponds to introducing Dirichlet boundary condition (e.g., ref. 18, Chapter VII]).

Let us define for any bounded region $\Lambda\subset\mathbf{Z}^\nu$ the partition function $\mathbf{Z}_\Lambda$ and the local Gibbs state $\omega_\Lambda(A)$ by the following formulas

$$\begin{aligned} Z_\Lambda&=Tr_{\mathscr{H}_\Lambda}(e^{-\beta H_\Lambda})\\ \omega_\Lambda(A)&=Z_\Lambda^{-1}\,Tr_{\mathscr{H}_\Lambda}(Ae^{-\beta H_\Lambda})\end{aligned} \qquad (2.6)$$

where β is inverse temperature and A belongs to the algebra of the local observables (see, e.g., ref. 19).

We define $d\mu_\Lambda^\beta(\omega_\Lambda)$ as a measure on the β-periodic continuous path space $\Omega_\Lambda^\beta=C([0,\beta];\mathbf{R})$, with the help of the conditional Wiener measures:

$$d\mu_I^\beta(\omega_\Lambda)=P_\Lambda^\beta(x_\Lambda,x_\Lambda;d\omega_\Lambda)\,dx_\Lambda=\prod_{i\in\Lambda}P^\beta(x_i,x_i;d\omega_i)\,dx_i \qquad (2.7)$$

with the condition $\omega_i(0)=\omega_i(\beta)=x_i$. Then (see refs. 13, 16, 17, 20 for details

$$Z_\Lambda=\int d\mu_\Lambda^\beta(\omega_\Lambda)\,e^{-V(\omega_\Lambda)} \qquad (2.8)$$

$$V(\omega_\Lambda)=U(\omega_\Lambda)+W(\omega_\Lambda) \qquad (2.9)$$

$$U(\omega_\Lambda)=\sum_{i\in\Lambda}[\lambda(\omega_i^2-\lambda^{-1})^2+\tfrac{1}{2}\lambda\omega_i^2]$$

$$W(\omega_\Lambda)=\tfrac{1}{2}\sum_{(i,j)\in\Lambda}d_{ij}(\omega_i-\omega_j)^2$$

with

$$f(\omega) = \int_0^\beta d\tau\, f(\omega(\tau)).$$

3. THE EXISTENCE OF PHASE TRANSITIONS. PEIERLS TYPE ARGUMENTS

To use a Peierls type argument we should have some kind of a "collective spin variable" which takes two values $+1$ or -1 (or a "many-component spin" in the frameworks of Fröhlich[11] or, in another context, of Pirogov-Sinai[21]). Following the idea of Glimm-Jaffe-Spencer[9] we define our "spin variable" as the sign of the mean value of the Wiener loop $\omega(\tau)$, $0 \leqslant \tau \leqslant \beta$, i.e.,

$$\xi_i = \frac{1}{\beta} \int_0^\beta \omega_i(\tau)\, d\tau, \qquad \text{sign}(\xi_i) = \sigma_i, \qquad i \in \mathbf{Z}^\nu \tag{3.1}$$

As a long-range order parameter we shall consider the two-point correlation function $\langle \sigma_i \sigma_j \rangle_\Lambda$, where $\langle \cdot \rangle_\Lambda$ is the mean value that is defined by the following expression:

$$\langle \cdot \rangle_\Lambda = \int \cdot\, d\rho_\Lambda^\beta(\omega_\Lambda) = Z_\Lambda^{-1} \int \cdot\, d\mu_\Lambda^\beta(\omega_\Lambda) \tag{3.2}$$

From Lemma 3.3.2 of ref. 16 we have the existence of the limit correlation function

$$\langle \sigma_i \sigma_j \rangle = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \langle \sigma_i \sigma_j \rangle_\Lambda \tag{3.3}$$

Let us also remark that because of symmetry $\omega \to -\omega$ we have $\langle \sigma_j \rangle = 0$. Now the main statement of this paper is:

Theorem 3.1. Let the limit measure $d\nu^\beta(\omega)$ exist for the system which is determined by Eqs. (2.3)–(2.5). Then for some fined β, sufficiently small $\lambda > 0$ and arbitrary $j, k \in \mathbf{Z}^\nu$

$$\langle \sigma_j \sigma_k \rangle > \tfrac{1}{2}$$

which implies the existence of long-range order behavior and of at least two ground states.

Proof. We divide the space $\mathbf{R}^\nu$ (which contains the lattice $\mathbf{Z}^\nu$) to ν-dimensional unit cubes which have their centers at the sites of the lattice $\mathbf{Z}^\nu$. There exists thus a correspondence between sites and cubes: $\omega_j(\tau) \equiv \omega_{\Delta_j}(\tau)$, $j \in \Delta_j$.

Let $\bar{\Delta}$ be the set of all cubes Δ. In every $j \in \mathbf{Z}^\nu$ (or $\Delta_j \in \bar{\Delta}$) some path $\omega_j(\tau)$ $(0 \leqslant \tau \leqslant \beta$, $\omega_j(0) = \omega_j(\beta))$, its mean value ξ_j and "spin" s_j are defined. The set of all fixed ξ_j, $j \in \mathbf{Z}^\nu$ defines a configuration σ. The boundary Γ of the configuration σ is the set of those faces of the nearest neighbours Δ and Δ' for which $\sigma(\Delta) = -\sigma(\Delta')$. It is clear that Γ consists of different closed contours ($(\nu-1)$-dimensional surfaces). Let $N(\Gamma)$ be the set of those nearest cubes, the faces of which form Γ. We define

$$\mathscr{X}_\pm(\Delta) = \tfrac{1}{2}(1 \pm \sigma_\Delta)$$

Then

$$\langle \sigma_\Delta \sigma_{\Delta'} \rangle = 1 - 4\langle \mathscr{X}_+(\Delta)\, \mathscr{X}_-(\Delta') \rangle \tag{3.4}$$

So, to prove Theorem 3.1 we should prove that for sufficiently small λ the mean value of $\mathscr{X}_+ \mathscr{X}_-$ on r.h.s. of (3.4) is less than 1/8, independently of the choice of Δ and Δ'.

By definition of $\mathscr{X}_\pm(\Delta)$ for any Δ we have

$$\mathscr{X}_+(\Delta) + \mathscr{X}_-(\Delta) = 1$$

and for any $\Lambda_0 \subset \mathbf{Z}^\nu$

$$\prod_{\Delta \subset \Lambda_0} (\mathscr{X}_+(\Delta) + \mathscr{X}_-(\Delta)) = 1 \tag{3.5}$$

Let us write

$$\langle \mathscr{X}_+(\Delta_i)\, \mathscr{X}_-(\Delta_j) \rangle = \langle \mathscr{X}_+(\Delta_i)\, \mathscr{X}_-(\Delta_j) \cdot 1 \rangle$$

Using (3.5) and taking into account that $\mathscr{X}_+(\Delta)\, \mathscr{X}_-(\Delta) = 0$ and $\mathscr{X}^2_\pm(\Delta) = \mathscr{X}_\pm(\Delta)$, we pass from the product to the sum over all possible configurations σ: $\Delta \to \pm 1$

$$\langle \mathscr{X}_+(\Delta_i)\, \mathscr{X}_-(\Delta_j) \rangle = \sum_\sigma \left\langle \mathscr{X}_+(\Delta_i)\, \mathscr{X}_-(\Delta_j) \prod_{\Delta \subset \Lambda_0,\, \Delta \neq \Delta_i,\, \Delta_j} \mathscr{X}_\sigma(\Delta) \right\rangle \tag{3.6}$$

Every configuration leads to the decomposition of Λ_0 "+" and "−" connected components X_i. The boundary Γ is the union of all connected contours: $\Gamma = \bigcup_i \gamma_i$. Let Δ_i belong to the some connected component $X_k(+)$ and Δ_j belong to the some connected component $X_l(-)$. Then there exists some contour γ that divides Δ_i and Δ_j such that the sum over all possible configuration in (3.6) can be represented as the sum over all possible contours γ (which divide Δ_i and Δ_j) and the sum over all possible configurations for which the given contour γ does not change:

$$\langle \mathscr{X}_+(\Delta_i)\,\mathscr{X}_-(\Delta_j)\rangle = \sum_{\gamma}\ \sum_{c\,:\,\gamma(c)=\gamma} \left\langle \mathscr{X}_+(\Delta_i)\,\mathscr{X}_-(\Delta_j) \prod_{\Delta\subset\Lambda_0,\,\Delta\neq\Delta_i,\,\Delta_j} \mathscr{X}_\sigma(\Delta)\right\rangle \tag{3.7}$$

Now for every contour γ there exists some set of cubes N_γ which consists of nearest neighbours (Δ', Δ'') such that

$$\bigcup_{\Delta',\,\Delta''\subset N_\gamma} (\Delta'\cap\Delta'') = \gamma$$

Moreover, for any given γ, the sum over $\sigma : \gamma(\sigma) = \gamma$ runs over all different configurations in $\Lambda_0\backslash N_\gamma$ but with the restriction that $\sigma(\Delta_i) = +1$, and $\sigma(\Delta_j) = -1$. If we remove this restriction the sum becomes greater and using again identity (3.5) for $\Lambda_0\backslash N_\gamma$ we get the following inequality:

$$\langle \mathscr{X}_+(\Delta_i)\,\mathscr{X}_-(\Delta_j)\rangle \leqslant \sum_{\gamma} \left\langle \prod_{(\Delta',\,\Delta'')\in N_\gamma} \mathscr{X}_+(\Delta')\,\mathscr{X}_-(\Delta'')\right\rangle \tag{3.8}$$

Now we derive an extension of Peierls inequality, which we formulate in the following

Theorem 3.2. For sufficiently small λ and any set of pairs of the neighbour cubes $N(\Gamma)$ the following inequality holds:

$$\left\langle \prod_{(\Delta,\,\Delta')\in N(\Gamma)} \mathscr{X}_+(\Delta)\,\mathscr{X}_-(\Delta)\right\rangle \leqslant \exp\{-const\,\lambda^{-1/2}\,|N(\Gamma)|\} \tag{3.9}$$

Using Theorem 3.2 and the fact that

$$|\gamma| \leqslant N_\gamma \leqslant 2\,|\gamma|$$

we obtain

$$\langle \mathscr{X}_+(\Delta_i)\,\mathscr{X}_-(\Delta_j)\rangle \leqslant \sum_{\gamma} e^{-const\,\lambda^{-1/2}\,|\gamma|} \tag{3.10}$$

Taking into account that $|\gamma| \geqslant 2\nu$ and the number of contours with $|\gamma| = n$ is not greater that $2\nu(3(2^{\nu-1}-1))^n\,(n/2\nu)^{\nu/(\nu-1)}$, it is easy to obtain the proof of Theorem 3.1 (following ref. 9).

Proof of Theorem 3.2. For every pair $(\Delta, \Delta') \in N(\Gamma)$ we make the decomposition:

$$\mathscr{X}_+(\Delta)\,\mathscr{X}_-(\Delta') = (\mathscr{X}^s_+(\Delta) + \mathscr{X}^l_+(\Delta))(\mathscr{X}^s_-(\Delta') + \mathscr{X}^l_-(\Delta')) \qquad (3.11)$$

where

$$\mathscr{X}^s_+(\Delta) = \mathscr{X}_{(0,\,(1/2)\,\lambda^{-1/2})}(\Delta),\ \xi_\Delta \in (0, (1/2)\,\lambda^{-1/2})$$
$$\mathscr{X}^l_+(\Delta) = \mathscr{X}_{((1/2)\,\lambda^{-1/2},\,\infty)}(\Delta),\ \xi_\Delta \in ((1/2)\,\lambda^{-1/2}, \infty)$$
$$\mathscr{X}^s_-(\Delta) = \mathscr{X}_{(-(1/2)\,\lambda^{-1/2},\,0)}(\Delta),\ \xi_\Delta \in (-(1/2)\,\lambda^{-1/2}, 0)$$
$$\mathscr{X}^l_-(\Delta) = \mathscr{X}_{(-\infty,\,-(1/2)\,\lambda^{-1/2})}(\Delta),\ \xi_\Delta \in (-\infty, -(1/2)\,\lambda^{-1/2})$$

Let us estimate all four terms which appears on the r.h.s. of (3.11) after carrying through the multiplication.

(a) $\rho_l \equiv \mathscr{X}^l_+(\Delta)\,\mathscr{X}^l_-(\Delta')$ is equal to 1 (i.e., is not equal to 0) if $\xi_\Delta \geqslant (1/2)\,\lambda^{-1/2}$ and $\xi_{\Delta'} \leqslant -(1/2)\,\lambda^{-1/2}$. Then $\xi_\Delta - \xi_{\Delta'} \geqslant \lambda^{-1/2}$, and for any integer even M:

$$[\lambda(\xi_\Delta - \xi_{\Delta'})^2]^M \geqslant 1$$

Because of $\rho_l(\Delta, \Delta') \leqslant 1$ we obtain the following estimate:

$$\rho_l(\Delta, \Delta') \leqslant [\lambda(\xi_\Delta - \xi_{\Delta'})^2]^M = \lambda^{M/2} \frac{d^M}{d\varepsilon_l^M} e^{\varepsilon_l \lambda^{1/2} Q_l(\Delta, \Delta')} \bigg|_{\varepsilon_l = 0}$$

where

$$Q_l(\Delta, \Delta') = (\xi_\Delta - \xi_{\Delta'})^2$$

(b) $\mathscr{X}^s_+(\Delta)\,\mathscr{X}^s_-(\Delta') \leqslant \mathscr{X}^s_+(\Delta) \equiv \rho^+_s(\Delta)$.

(c) $\mathscr{X}^s_+(\Delta)\,\mathscr{X}^l_-(\Delta') \leqslant \mathscr{X}^s_+(\Delta) \equiv \rho^+_s(\Delta)$.

(d) $\mathscr{X}^l_+(\Delta)\,\mathscr{X}^s_-(\Delta') \leqslant \mathscr{X}^s_+(\Delta') \equiv \rho^-_s(\Delta')$.

For the cases (b)–(d) we can write general estimates using the condition $|\xi_\Delta| \leqslant (1/2)\,\lambda^{-1/2}$:

$$\rho^\#(\Delta) \leqslant \frac{4}{3}(1 - \lambda\xi_\Delta^2), \qquad \rho^\#(\Delta) = \rho^\pm_s(\Delta)$$

$$\rho^\#(\Delta) \leqslant \left[\frac{4}{3}(1 - \lambda\xi_\Delta^2)\right]^M = \lambda^{M/2} \frac{d^M}{d\varepsilon_s^M} e^{\varepsilon_s \lambda^{1/2} Q_s(\Delta)} \bigg|_{\varepsilon_s = 0}$$

where

$$Q_s(\Delta) = (\xi_\Delta^2 - \lambda^{-1})$$

Then

$$\mathscr{X}_+(\Delta)\,\mathscr{X}_-(\Delta') \leqslant 2^M \lambda^{M/2} \left(\frac{d^M}{d\varepsilon_l^M} e^{\varepsilon_l \lambda^{1/2} Q_l(\Delta, \Delta')} \Big|_{\varepsilon_l = 0} + \frac{d^M}{d\varepsilon_s^M} e^{\varepsilon_s \lambda^{1/2} Q_s(\Delta)} \Big|_{\varepsilon_s = 0} \right.$$
$$\left. + \frac{d^M}{d\varepsilon_s^M} e^{\varepsilon_s \lambda^{1/2} Q_s(\Delta')} \Big|_{\varepsilon_s = 0} + \frac{d^M}{d\varepsilon_s^M} e^{\varepsilon_s \lambda^{1/2} Q_s(\Delta)} \Big|_{\varepsilon_s = 0} \right)$$

The end of the proof of Theorem 3.2 is analogous to ref. 9 (see also ref. 14). We apply namely the Cauchy inequality for derivatives (in every variable). Using the boundedness of the mean value we obtain

$$\langle e^{\lambda^{1/2} Q(\varepsilon, B)} \rangle = \lim_{\Lambda \nearrow \mathbf{Z}^\nu} \int e^{\lambda^{1/2} Q(\varepsilon, B)}\, d\nu_\Lambda^\beta(\omega_\Lambda) \tag{3.12}$$

with

$$Q(\varepsilon, B) = \sum_{\langle i, j \rangle \subset B} Q_l(\varepsilon_{ij}, \Delta_i, \Delta_j) + \sum_{i \in B} Q_s(\varepsilon_i, \Delta_i) \tag{3.13}$$

where the first sum is the sum over nearest neighbour pairs and $\lambda^{1/2} |\varepsilon| \leqslant 1/4d$ (here ε is one of ε_{ij} or ε_i). B is any finite set $B \subset \Lambda \subset \mathbf{Z}^\nu$. The latter statement and the proof of Theorem 3.2 are a consequence of the main technical lemma:

Lemma 3.1. For any region $B \subset \Lambda$, $|B| < \infty$ there exist a constant c which does not depend on B, Λ and λ, such that for sufficiently small λ

$$\rho^\Lambda(Q, (\varepsilon, B)) \equiv Z_\Lambda^{-1} \int d\mu(\omega_\Lambda) \exp\{ -V(\omega_\Lambda) + \lambda^{1/2} Q(\varepsilon, B)\} \leqslant e^{c|B|} \tag{3.14}$$

where $Q(B)$ is a polynomial of the second degree with respect to the variables $\omega_j(\tau)$, $j \in B$, defined by (3.13).

Remark 3.1. Lemma 3.1 is a partial case of Lemma 3.2.1 in ref. 16. The difference between these lemmas lies in the fact that to prove the main result of the present paper we need more information on the detailed dependence of the constant c on parameter λ. The main technical trick, which allows us to avoid the occurence of large terms of the form $\lambda^{-1/2} |B|$, consists in exploiting the transformation properties of Wiener integrals under translations.

Remark 3.2. Note that $c \sim \beta^{-\alpha}, \alpha > 0$. So, if λ is fixed, making β small we can destroy the long-range order behaviour. ▮

4. THE BOUNDEDNESS OF QUANTUM STATES AND THE PROOF OF LEMMA 3.1

For our proof of the bound (3.14) and of Theorems 3.1 and 3.2 we use methods of refs. 15, 16 (see also ref. 22 for the case of quantum systems) and refs. 9, 14. Let us remark that in (3.14) we obtain a better constant than in work of Park (ref. 16, Proposition 2.1). The constant on the right hand side of (3.14) is proportional to $\lambda^{-1/2}$, which makes it possible to obtain from it a proof of Theorems 3.1 and 3.2. In addition we avoid the use of reflection positivity.

We start by recalling some results which originated from ref. 15 (Proposition 2.1). For given $\alpha > 0$, we can choose an integer $p_0 > 0$ and for each $j \geqslant p_0$ an integer $l_j > 0$ such that

$$\alpha < \frac{l_{j+1}}{l_j} - 1 < 3\alpha$$

We use the notation

$$[j] = \{i \in \mathbf{Z}^\nu : |i| \leqslant l_j\}, \qquad [k \backslash j] = [k] \backslash [j]$$
$$V_j = (2l_j + 1)^\nu$$

Then the following lemma holds[15, 16] (see also ref. 23):

Lemma 4.1. Let $\varepsilon' > 0$, $c' \geqslant 0$, and let ψ be an increasing positive function on the positive integers such that

$$\sum_{i \in \mathbf{Z}^\nu} \psi(|i|)\, d_{ij} < \infty$$

If α is sufficiently small, one can choose an increasing sequence (ψ_j) such that $\psi_j \geqslant 1$, $\psi_j \to \infty$, and one can fix $p > p_0$ so that the following is true.

Suppose that there exists q such that $q \geqslant p$ and q is the largest integer for which

$$\sum_{i \in [q]} n(i)^2 \geqslant \psi_q V_q$$

$n(\cdot)$ being a function from $\mathbf{Z}^\nu$ to the positive integers. Then

$$\sum_{i \in [q+1]} c' + \sum_{i \in [q+1]} \sum_{j \notin [q+1]} d_{ij} [\tfrac{1}{2} n(i)^2 + \tfrac{1}{2} n(j)^2] \leqslant \varepsilon' \sum_{i \in [q+1]} n(i)^2$$

Remark 4.1. It is important to realize (see refs. 15, 16) that

$$\psi_p V_p \sim \left(\frac{c'}{\varepsilon'}\right)^{K_0}$$

where K_0 depends only on ν and power of decreasing of the function $d_{|i-j|}$.

To apply Lemma 4.1 to our situation let us divide the space of all Wiener paths Ω_Λ^β into a countable sequence of mutually disjont sets

$$\Omega_\Lambda^\beta = Q_0 \cup \bigcup_{q \geqslant p} Q_q$$

$$Q_0 = \left\{ \omega \in \Omega_\Lambda^\beta : \sum_{i \in [q]} \omega_i^2 < \psi_q V_q \; \forall q \geqslant p \right\} \tag{4.1}$$

$$Q_q = \left\{ \omega \in \Omega_\Lambda^\beta : \sum_{i \in [q]} \omega_i^2 \geqslant \psi_q V_q \text{ and } \sum_{i \in [l]} \omega_i^2 < \psi_l V_l \text{ for } l \geqslant +1 \right\}$$

Let $Q(\varepsilon, B)$ be defined by

$$Q(\varepsilon, B) = \sum_{\mu=1}^{3} Q_\mu(\varepsilon, B) \tag{4.2}$$

with

$$Q_1(\varepsilon, B) = \sum_{j \in B} \varepsilon_j \frac{1}{\beta} \int_0^\beta (\omega_j(\tau)^2 - \lambda^{-1})\, d\tau$$

$$Q_2(\varepsilon, B) = \sum_{j \in B} \varepsilon_j \left[\left(\frac{1}{\beta} \int_0^\beta \omega_j(\tau)\, d\tau \right)^2 - \frac{1}{\beta} \int_0^\beta \omega_j(\tau)^2\, d\tau \right]$$

$$Q_3(\varepsilon, B) = \sum_{\langle i, j \rangle} \varepsilon_{ij} \left(\frac{1}{\beta} \int_0^\beta (\omega_i(\tau) - \omega_j(\tau))\, d\tau \right)^2$$

Let us remind that using Cauchy's formula in the proof of Lemma 4.2 it is sufficiently to consider the case of real ε with $|\varepsilon| \leqslant 1$. Now we apply two times the Schwarz inequality to the quantity $\rho^\Lambda(\lambda^{1/2} Q(B))$ (on the left side of the inequality (3.14)) with respect to the measure

$$Z_\Lambda^{-1} e^{-V(\omega_\Lambda)}\, d\mu(\omega_\Lambda)$$

Then we obtain

$$\rho^{\Lambda}(Q(\varepsilon, B)) \leqslant \rho^{\Lambda}(2Q_1(\varepsilon, B))^{1/2}\, \rho^{\Lambda}(4Q_2(\varepsilon, B))^{1/4}\, \rho^{\Lambda}(4Q_3(\varepsilon, B))^{1/4} \tag{4.3}$$

Before estimating $\rho^{\Lambda}(2Q_1(\varepsilon, B))$ we perform the following translations in the integral with respect to the measure $d\mu(\omega_\Lambda)$

$$\omega_j(\tau) \to \omega_j(\tau) + \lambda^{-1/2} \tag{4.4}$$

to avoid the appearence of the terms which are proportional to $\lambda^{-1/2}\,|\Lambda|$. Then (see ref. 24)

$$\rho^{\Lambda}(2Q_1(\varepsilon, B)) = Z_\Lambda^{-1} \int d\mu(\omega_\Lambda)\, e^{-V'(\omega_\Lambda) + 2Q_1'(\varepsilon, B)} \tag{4.5}$$

with

$$\begin{aligned} V'(\omega_\Lambda) &= U'(\omega_\Lambda) + W(\omega_\Lambda) \\ U'(\omega_\Lambda) &= \sum_{i \in \Lambda} \left[(\lambda^{1/2}\omega_i^2 + 2\omega_i)^2 + \tfrac{1}{2}(\lambda^{1/2}\omega_i + 1)^2\right] \\ Q_1'(\varepsilon, B) &= \sum_{i \in B} \varepsilon_i(\lambda^{1/2}\omega_i^2 + 2\omega_i) \end{aligned} \tag{4.6}$$

As it was shown in the papers,[16, 17] the proof of the Lemma 3.1 is a consequence of the following proposition.

Proposition 4.1. Assume that $B \cap [p] \neq \varnothing$ and $i \in B \cap [p]$. Then there exist constants c_1 and γ, $0 < \gamma < 1$ such that

$$\rho^{\Lambda}(2Q_1(\varepsilon, B)) \leqslant e^{c_1}\rho^{\Lambda}(2Q_1(\varepsilon, B\backslash\{i\})) + \sum_{q \geqslant p} \gamma^{V_q}\rho^{\Lambda}(2Q_1(\varepsilon, B\backslash[q+1])) \tag{4.7}$$

Lemma 3.1 for $\rho^{\Lambda}(2Q_1(\varepsilon, B))$ follows from the above proposition using an induction with respect to card(B).

Proof of Proposition 4.1. According to the decomposition (4.1) we have

$$\rho^{\Lambda}(2Q_1(\varepsilon, B)) = \rho_0^{\Lambda} + \sum_{q \geqslant p} \rho_q^{\Lambda} \tag{4.8}$$

Let $i \in B \cup [p] \neq \varnothing$ for some large p. Then we write

$$\rho_0^A = Z_A^{-1} \int_{Q_0} d\mu(\omega_{A\backslash\{i\}}) \exp\{-V'(\omega_{A\backslash\{i\}}) + 2Q_1'(\varepsilon, B\backslash\{i\})\}$$
$$\times \int_{Q_0} d\mu(\omega_i) \exp\left\{-U'(\omega_i) + 2Q_1'(\varepsilon_i, i) - \tfrac{1}{4} \sum_{j \in A\backslash\{i\}} d_{ij}(\omega_i - \omega_j)^2\right\} \tag{4.9}$$

Let

$$Q_{R_0} = \{\omega'(\tau) \mid |\omega'(\tau)| \leqslant R_0, 0 \leqslant \tau \leqslant \beta\}$$

Then it is easy to obtain the following estimate

$$K_1 \int_{Q_{R_0}} d\mu(\omega_i')\, e^{-U'(\omega_i')} \geqslant 1 \tag{4.10}$$

with a constant K_1 that depends only on R_0 and β. Analogously we obtain the following estimate

$$-\tfrac{1}{4} \sum_{j \in A\backslash\{i\}} d_{ij}(\omega_i - \omega_j)^2 \leqslant -\tfrac{1}{4} \sum_{j \in A\backslash\{i\}} d_{ij}(\omega_i' - \omega_j)^2 + D \tag{4.11}$$

with

$$D = \tfrac{1}{2} d_0 \psi_p V_p + \tfrac{1}{2}(1 + 3\alpha)^{\nu+1} \sum_{l \geqslant l_p} [d_l - d_{l+1}]\, \psi(l)(2l+1)^\nu$$

(see, for instance, refs. 15, 16 or ref. 23, Chapter XVI). Using the Goldon--Tompson inequality, one gets

$$\int_{Q_0} d\mu(\omega_i)\, e^{-U'(\omega_i) + 2Q_1'(i)}$$
$$\leqslant \int_{\Omega} d\mu(\omega_i)\, e^{-U'(\omega_i) + 2Q_1'(i)}$$
$$\leqslant (2\pi\beta)^{-1/2} \int_{\mathbf{R}} dx \exp\{-\beta(\lambda^{1/2}x^2 + 2x)^2$$
$$-\tfrac{1}{2}\beta(\lambda^{1/2}x + 1)^2 + 2\varepsilon(\lambda^{1/2}x^2 + 2x)\} \leqslant K_2 \tag{4.12}$$

with K_2 which does not depend on λ and $|\varepsilon| = 1$. So inserting an integration with respect to $d\mu(\omega_i)$ on the right hand side of (4.9), we obtain with help of (4.10) and using (4.11) and (4.12)

$$\rho_0^\Lambda \leqslant K_1 K_2 e^{\beta D} \rho^\Lambda(2Q_1(B\backslash\{i\})) \tag{4.13}$$

Using the same procedure we get

$$(K_1)^{|[q+1]\cap\Lambda|} \int_{Q_{R_0}^{[q+1]\cap\Lambda}} d\mu(\omega'_{[q+1]\cap\Lambda})\, e^{-V'(\omega'_{[q+1]\cap\Lambda})} \geqslant 1 \tag{4.14}$$

Then

$$\begin{aligned} &-W(\omega_{[q+1]\cap\Lambda}, \omega_{\Lambda\backslash[q+1]}) \\ &\quad\leqslant -W(\omega'_{[q+1]\cap\Lambda}, \omega_{\Lambda\backslash[q+1]}) + \sum_{i\in[q+1]} \sum_{j\in\Lambda\backslash[q+1]} \frac{d_{ij}}{2}(\omega_i'^2 + \omega_j^2) \\ &\quad\leqslant -W(\omega'_{[q+1]\cap\Lambda}, \omega_{\Lambda\backslash[q+1]}) + \varepsilon' \sum_{i\in[q+1]} \omega_i^2 - c' V_{q+1} \end{aligned} \tag{4.15}$$

In the latter inequality we used Lemma 4.1 with c' and ε' which we shall choose latter. Because

$$\exp\{-W(\omega_{[q+1]\cap\Lambda})\} \leqslant 1$$

the estimate of the integral

$$\int_{Q_q} d\mu(\omega_{[q+1]\cap\Lambda}) \exp\{-V'(\omega_{[q+1]\cap\Lambda}) + Q_1'([q+1]\cap B) + \varepsilon'\omega^2_{[q+1]\cap\Lambda}\}$$

is reduced to the estimate of the integral

$$I(\varepsilon') = \int_\Omega d\mu(\omega) \exp\{-U'(\omega) + 2Q_1'(\omega) + \varepsilon'\omega^2\}$$

Choosing $\varepsilon' < 1/4$, we get

$$I(\varepsilon') \leqslant K_3$$

with K_3 that does not depend on λ. So

$$\rho_q^\Lambda \leqslant (K_1 K_3)^{V_{q+1}} e^{-c' V_{q+1}} \rho^\Lambda(2Q(B\backslash[q+1])) \tag{4.16}$$

From (4.13) and (4.16) we get (4.7) with

$$c_1 = \beta D + \log[K_1 K_2]$$
$$\gamma = K_1' K_3 e^{-c'}$$

for sufficiently large c' in Lemma 4.1. So we get the proof of Lemma 3.1 for $\rho^A(2Q_1(\varepsilon, B))$. The estimate of $\rho^A(4Q_2(\varepsilon, B))$ can be done in the same way if $Q_2(j)$ is represented,in the form

$$Q_2(j) = -\varepsilon_j \frac{1}{\beta} \int_0^\beta (P\omega_j)^2(\tau)\, d\tau$$

where P is the operator of orthoprojection in $L^2[0, \beta]$:

$$(P\omega)(\tau) = \omega(\tau) - \frac{1}{\beta} \int_0^\beta \omega(\tau)\, d\tau$$

The translation (4.4) does not change the expression for Q_2 and taking into account that P has unit norm, we get

$$Q_2'(j) \leqslant \lambda^{1/2} \frac{1}{\beta} \int_0^\beta \omega_j^2(\tau)\, d\tau$$

The term Q_3 also is not changed under the translation (4.4) and using the lower bound of (2.5) the third factor in (4.3) can be estimated in the same way. Really, the only difference is that instead of one site $i \in B$ we should take two nearest sites $i, j \in B$, $(|i-j| = 1)$ and therefore estimate the following integral

$$\int_{Q_0} d\mu(\omega_i)\, d\mu(\omega_j)\, e^{-U'(\omega_i) - U'(\omega_j) - d_{ij}(\omega_i - \omega_j)^2 + 4Q_3'(\varepsilon, \langle i, j \rangle)}$$

The application (2.5) leads again to (4.12) for sufficiently small λ. ∎

5. CONCLUSION

Our results imply (see ref. 14) that for a fixed temperature and mass of the particles we can choose λ sufficiently small (i.e. increasing the depth of the minima of the interaction U and the distances between them) and obtain at least two pure different phases.

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REFERENCES

1. J. Fröhlich, B. Simon, and T. Spencer, Infrared bounds, phase transitions and continuous symmetry breaking, *Commun. Math. Phys.* **50**:79 (1976).
2. F. Dyson, E. Lieb, and B. Simon, Phase transitions in quantum spin systems with isotropic and nonisotropic interactions, *J. Stat. Phys.* **18**:335 (1978).
3. W. Driesler, L. Landau, and J. F. Perez, Estimates of critical lengths and critical temperatures for classical and quantum lattice systems, *J. Stat. Phys.* **20**:123 (1979).
4. L. A. Pastur and B. A. Khoruzhenko, Phase transitions in quantum models of rotators and ferroelectrics, *Theor. Math. Phys.* **73**:1094 (1987).
5. V. S. Barbulyak and Yu. G. Kondratiev, Functional integrals and quantum lattice systems: Phase transitions, *Reports Nat. Acad. Sci. of Ukraine* **10**:19 (1991).
6. V. S. Barbulyak and Yu. G. Kondratiev, The semiclassical limit for the Schrödinger operator and phase transitions in quantum statistical mechanics, *Funct. Anal. Appl.* **26**:124 (1992).
7. Yu. G. Kondratiev, Phase transitions in quantum models of ferroelectrics, in *Stochastic Processes, Physics and Geometry II* (World Scientific, Singapore/New Jersey, 1994), pp. 465–475.
8. A. Yu. Kondratiev, Phase transitions in quantum models of ferromagnetics. Approach with using the functional integrals, *Reports Nat. Acad. Sci. of Ukraine* **6** (1996).
9. J. Glimm, A. Jaffe, and T. Spencer, Phase transitions for φ_2^4 quantum fields, *Common. Math. Phys.* **45**:203 (1975).
10. R. Peierls, On Ising's model of ferromagnetics, *Proc. Cambridge Phil. Soc.* **32**:477 (1936).
11. J. Fröhlich, Phase transitions, Goldstone bosons and topological superselections rules, *Acta Physica Austriaca, Suppl. XV*, 133–269 (1976).
12. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
13. S. Albeverio and R. Hoegh-Krohn, Homogeneous random fields and statistical mechanics, *J. Funct. Anal.* **19**:242 (1975).
14. J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View* (Springer-Verlag, New York, 1982).
15. D. W. Ruelle, Superstable interaction in classical statistical mechanics, *Common. Math. Phys.* **18**:127 (1970).
16. Y. M. Park, Quantum statistical mechanics of unbounded continuous spin systems, *J. Korean Math. Soc.* **22**:43 (1985).
17. Y. M. Park and H. J. Yoo, Uniqueness and clustering properties of Gibbs states for classical and quantum unbounded spin systems, *J. Stat. Phys.* **80**:223 (1995).

18. B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory* (Princeton, New Jersey, 1974).
19. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I, II* (Springer-Verlag, New York/Heidelberg/Berlin, 1981).
20. V. S. Barbulyak and Yu. G. Kondratiev, Functional integrals and quantum lattice systems: Existence of Gibbs states, *Reports Nat. Acad. Sci. of Ukraine* **8**:31 (1991).
21. S. Pirogov and Ya. G. Sinai, Phase diagrams of classical lattice systems, *Theor. Math. Phys.* **25**:1185 (1975); **26**:39 (1976).
22. R. Esposito, F. Nicoló, and M. Pulvirenti, Superstable interactions in quantum statistical mechanics: Maxwell–Boltzmann statistics, *Ann. Inst. Henri Poincaré* **XXXIV**:127 (1982).
23. D. Ya. Petrina, V. I. Gerasimenko, and P. V. Malyshev, *Mathematical Foundations of Classical Statistical Mechanics: Continuous Systems* (Gordon & Breach, London, 1989).
24. J. Ginibre, Some applications of functional integration in statistical mechanics, in *Statistical Mechanics and Quantum Field Theory*, C. De Witt and R. Stora, eds. (New York, Gordon & Breach).